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Electron self-mass in the semiclassical limit

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Abstract. We have calculated the electron self-mass in quantum electrodynamics which is valid to all orders in e^2 but only to first order in an expansion in powers of $\hbar$. Our value is given by

$$\delta m = \frac{e^2}{2r_0c^2}\left(1+\frac{2\hbar}{\pi mcr_0}\right),$$

where r_0 is the cutoff radius.

The self-mass

$$\delta m_{\mathrm{Q}} = \frac{e^2}{4\pi\hbar c}\frac{3m}{2\pi}\left(\ln\frac{\hbar}{r_0mc}+\frac{1}{4}\right) \tag{1}$$

of the electron in quantum electrodynamics evaluated in the second-order perturbation theory is logarithmically divergent in the limit of the cutoff radius $r_0\to 0$ and proportional to its mass, in contrast to the classical value

$$\delta m_{\mathrm{c}} = \frac{e^2}{2r_0c^2} \tag{2}$$

which is linearly divergent in the limit $r_0\to 0$ and independent of m. Ordinarily one would expect the classical result to follow from the corresponding quantum mechanical expression in the limit $\hbar\to 0$, but there is a case which seems to violate this principle (see, for instance, Schweber 1961). This puzzle has recently been resolved by Vilenkin and Femin (1975) who point out that whereas δm_{Q} given by equation (1) is the first term in a perturbation expansion (in powers of $e^2/\hbar c$) of the electron self-mass, δm_{c} given by equation (2) is exact to all orders in e^2. Further, since the expansion parameter diverges in the limit $\hbar\to 0$, one should not expect to obtain a correspondence between these two. On the other hand, there should exist such a correspondence between the exact self-mass in quantum and classical electrodynamics. These authors show that Dyson's expression for the exact quantum electrodynamic self-mass equals the classical value in the limit $\hbar\to 0$. It has been noted by these authors that this result is similar to the well known theorem of Thirring (1950) which states that the exact Compton amplitude equals the classical (Thomson) amplitude in the limit of zero photon momentum.

The aim of the present investigation is to calculate the semiclassical limit of the electron self-mass which is the first-order term in an expansion of the exact Dyson self-mass in powers of $\hbar$. This is similar in spirit to the work of Gell-Mann and

Goldberger (1956) and Low (1956) on the calculation of Compton amplitude to first order in photon momentum. We have succeeded in our endeavour and our result is given by:

$$\delta m_{sc} = \frac{e^2 \hbar}{\pi m r_0^2 c^3} \tag{3}$$

which is quadratically divergent in the limit $r_0 \to 0$. It may be noted that this result is quantum mechanical in the same sense as any WKB result and is exact to all orders in e^2, in contrast to the logarithmically divergent self-mass given by equation (1) which is valid to order $e^2/\hbar c$.

We start with the exact self-mass†

$$\delta m = \mathrm{i}\bar{u}(p)\Sigma(p)u(p)$$
$$= \frac{\mathrm{i}e_0^2}{4\pi^3 \hbar c} \int \mathrm{d}^4 q \bar{u}(p)(\gamma_\mu G(p-q)\Gamma_\nu(p-q, p)D_{\mu\nu}(q))u(p) \tag{4}$$

where $u(p)$ is the free Dirac spinor, e_0 is the unrenormalised charge and $G(p)$, $\Gamma_\nu(p, p')$ and $D_{\mu\nu}(q)$ are the exact electron propagator, vertex function and photon propagator respectively. These can be written in Lorentz covariant form as follows:

$$G(p) = -\frac{z_2(\gamma . p + \mathrm{i}mc)}{p^2 + m^2 c^2}(1 + C_1(p^2) + \gamma . pC_2(p^2)) = z_2 s(p)(1 + C_1(p^2) + \gamma . pC_2(p^2)) \tag{5a}$$

$$D_{\mu\nu}(q) = \frac{z_3 \delta_{\mu\nu}}{\mathrm{i}q^2}(1 + d(q^2)) \tag{5b}$$

$$\bar{u}(p')\Gamma_\mu(p', p)u(p) = \bar{u}(p')(\gamma_\mu F_1(q^2) + \mathrm{i}\sigma_{\mu\nu} q_\nu F_2(q^2))u(p) \tag{5c}$$

where

$$C_1(p^2 = -m^2 c^2) = C_2(p^2 = -m^2 c^2) = 0; \qquad d(q^2 = 0) = 0;$$
$$F_1(q^2 = 0) = 1; \qquad F_2(q^2 = 0) = \kappa/2m \tag{5d}$$

κ being the anomalous magnetic moment of the electron. To obtain expansion of the self-mass given by equation (4) in powers of $\hbar$ we shall, as in Vilenkin and Femin (1975), introduce the propagation vector $K = q/\hbar$ of the photon and use a convergence factor‡ $(1 + \frac{1}{4} r_0^2 K^2)^{-2}$ in the photon propagator whence

$$\delta m = \frac{\mathrm{i}e_0^2}{4\pi^3 c} \int \frac{\mathrm{d}^4 K}{(1 + \frac{1}{4} r_0^2 K^2)^2}(\gamma_\mu \tilde{G}(p - K\hbar)\Gamma_\nu(p - K\hbar, p)\tilde{D}_{\mu\nu}(K\hbar))u(p) \tag{6}$$

where we have put

$$\tilde{G}(p) = \hbar G(p); \qquad \tilde{D}_{\mu\nu}(q) = \hbar^2 D_{\mu\nu}(q).$$

† We follow the notation and metric etc of Akhiezer and Berestetskii (1965) without setting $\hbar = c = 1$.

‡ Our convergence factor is different from that used in Vilenkin and Femin (1975) and has been so chosen as to ensure convergence of the integral for δm to order $\hbar$ and at the same time yield the result of Vilenkin and Femin in the classical limit.

We now expand $\tilde{G}(p-K\hbar)$, $\tilde{D}_{\mu\nu}(K\hbar)$ and $\Gamma_\nu(p-K\hbar, p)$ in powers of $\hbar$ in the following manner:

$$\tilde{G}(p-K\hbar)=\tilde{G}^{(0)}(p,K)+\hbar\tilde{G}^{(1)}(p,K)+\dots \tag{7a}$$

$$\tilde{D}_{\mu\nu}(K\hbar)=\delta_{\mu\nu}(\tilde{D}^{(0)}(K)+\hbar^2\tilde{D}^{(2)}(K)+\dots) \tag{7b}$$

$$\Gamma_\nu(p-K\hbar,p)=\Gamma_\nu^{(0)}+\hbar\Gamma_\nu^{(1)}+\dots. \tag{7c}$$

It has been shown in Vilenkin and Femin (1975) that:

$$\tilde{G}^{(0)}(p,K)=z_2\frac{\gamma\,.\,p+imc}{2p\,.\,K} \tag{8a}$$

$$\tilde{D}^{(0)}(K)=\frac{z_3}{\mathrm{i}K^2} \tag{8b}$$

$$\Gamma_\nu^{(0)}u=z_1^{-1}\gamma_\nu u. \tag{8c}$$

Substituting (7*a*), (7*b*), (7*c*) and (8*c*) in equation (6) and collecting terms of order $\hbar$ we find:

$$\delta m_{\mathrm{sc}}=\frac{\mathrm{i}e_0^2\hbar}{4\pi^3c}\int\frac{\mathrm{d}^4K}{(1+\frac{1}{4}r_0^2K^2)^2}\bar{u}(p)\gamma_\mu(\tilde{G}^{(1)}(p,K)\gamma_\mu z_1^{-1}+\tilde{G}^{(0)}(p,K)\Gamma_\mu^{(1)}(K))u(p)\tilde{D}^{(0)}(K). \tag{9}$$

We need to know $\tilde{G}^{(1)}(p,K)$ and $\Gamma_\mu^{(1)}(K)$ before we can evaluate δm_{sc} given by equation (9). Taylor's expansion of $\tilde{G}(p-K\hbar)$ with the explicit representation (5*a*) we obtain

$$\tilde{G}^{(1)}(p,K)=\frac{\gamma\,.\,p+\mathrm{i}mc}{2p\,.\,K}\Big(\frac{K^2}{2p\,.\,K}-2p\,.\,K(C_1'+\gamma\,.\,pC_2')\Big)-\frac{\gamma\,.\,K}{2p\,.\,K} \tag{10}$$

where the primes on C_1 and C_2 denote differentiation with respect to their arguments, i.e. p^2. Since, on account of the Dirac equation for $\bar{u}$, we have from (8*a*) and (10)

$$\bar{u}(p)\gamma_\mu\tilde{G}^{(0)}(p,K)=\frac{z_2p_\mu}{p\,.\,K}\bar{u}(p) \tag{11}$$

$$\bar{u}(p)\gamma_\mu\tilde{G}^{(1)}(p,K)=z_2\bar{u}(p)\Big(\frac{\gamma\,.\,K\gamma_\mu}{2p\,.\,K}-\frac{K_\mu}{p\,.\,K}+\frac{p_\mu K^2}{2(p\,.\,K)^2}-2p_\mu(C_1'+\mathrm{i}mcC_2')\Big), \tag{12}$$

equation (9) further simplifies to

$$\delta m_{\mathrm{sc}}=\frac{\mathrm{i}e_0^2\hbar z_2}{4\pi^3c}\int\frac{\mathrm{d}^4K}{(1+\frac{1}{4}r_0^2K^2)^2}\bar{u}(p)\Big[\Big(\frac{\gamma\,.\,K}{p\,.\,K}+\frac{\gamma\,.\,pK^2}{2(p\,.\,K)^2}$$
$$-2\gamma\,.\,p(C_1'+\mathrm{i}mcC_2')\Big)z_1^{-1}+\frac{\gamma\,.\,p}{p\,.\,K}\Gamma^{(1)}(K)\Big]u(p)\tilde{D}^{(0)}(K). \tag{13}$$

In the above equation we need to know $\bar{u}(p)\Gamma^{(1)}(K)u(p)$ and $(C_1'+\mathrm{i}mcC_2')$ to obtain the final expression for δm_{sc}. From (5*c*), we have

$$\bar{u}(p)\Gamma_\mu^{(1)}(K)u(p)=\lim_{K\to 0}\frac{\mathrm{i}\kappa}{2m}\bar{u}(p)\sigma_{\mu\nu}K_\nu u(p)=0. \tag{14}$$

To calculate $(C_1' + imcC_2')$ we make use of the relation

$$\frac{\partial G(p)}{\partial p_\mu} = G(p)\Gamma_\mu(p, p)G(p) \tag{15}$$

obtained from $(\partial/\partial p_\mu)(G(p)G^{-1}(p)) = 0$ and the Ward identity

$$\frac{\partial}{\partial p_\mu} G^{-1}(p) = -\Gamma_\mu(p, p).$$

Multiplying both sides of equation (15) on the left-hand side by $\bar{u}(p)s^{-1}(p)$ and on the right-hand side by $s^{-1}(p)u(p)$ and using the representation (5*a*), we get

$$C_1' + imcC_2' = 0. \tag{16}$$

Substituting equation (16) in equation (13) and making use of the identity $z_1 = z_2$, we obtain after simplification

$$\delta m_{sc} = \frac{ie^2}{4\pi^3 c^3} \int \frac{d^4K}{K^2(1+\frac{1}{4}r_0^2K^2)^2}\left(-\frac{1}{m} + \frac{mK^2}{2(p\,.\,K)^2}\right) \tag{17}$$

where $e = z_3^{1/2}e_0$ is the renormalised charge. Since the above expression is Lorentz-invariant, we evaluate it in the rest frame of the electron, i.e. $\boldsymbol{p} = 0$, and obtain

$$\delta m_{sc} = \frac{ie^2}{4\pi^3 c^3} \int \frac{d^4K}{K^2(1+\frac{1}{4}r_0^2K^2)^2}\left(-\frac{1}{m} + \frac{K^2}{4K_4^2 m}\right) = \frac{e^2\hbar}{\pi m r_0^2 c^3} \tag{18}$$

which is quadratically divergent in the limit $r_0 \to 0$. The total self-mass up to first order in $\hbar$ is thus

$$\delta m = \delta m_c + \delta m_{sc} = \frac{e^2}{2r_0^2c^2}\left(1 + \frac{2\hbar}{\pi m c r_0}\right). \tag{19}$$

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